

A TANNAKIAN CONTEXT FOR GALOIS

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ABSTRACT. Strong similarities have been long observed between the Galois (Categories Galoisiennes) and the Tannaka (Categories Tannakiennes) theories of representation of groups. In this paper we construct an explicit Tannakian context for Galois theory, and prove the equivalence between its fundamental theorems. Since the theorem is known for the Galois context, this brings, in particular, a proof of the fundamental (recognition) theorem for a Tannakian context different than the known cases, where it is assumed that the unit of the tensor product is an object of finite presentation.

INTRODUCTION. Strong similarities have been long observed between the Galois (Categories Galoisiennes) and the Tannaka (Categories Tannakiennes) theories of representation of groups. In this paper we construct an explicit Tannakian context for Galois theory, and prove the equivalence between its fundamental theorems. Since the theorem is known for the Galois context, this brings, in particular, a proof of the fundamental (recognition) theorem for a Tannakian context different than the known additive cases [5], [4], [2], or their generalization [8], where it is assumed that the unit of the tensor product is an object of finite presentation (that is, filtered colimits in the tensor category are constructed as in the category of sets).

Joyal-Tierney extension of Grothendieck Galois theory of progroups and Galois topoi [1], generalizes this theory, in particular, to arbitrary localic groups and pointed atomic topoi in [7] Theorem 1.

For Galois theory we follow Dubuc [3], where he develops *localic Galois theory* and makes an explicit construction of the localic group of automorphisms $Aut(F)$ of a set-valued functor $\mathcal{E} \xrightarrow{F} \mathcal{E}ns$, and of a lifting $\mathcal{E} \xrightarrow{\tilde{F}} \beta^{Aut(F)}$ into the topos of sets furnished with an action of the localic group (see 4.1). He proves in an elementary way¹ that when F is the inverse image of a point of an atomic topos, this lifting is an equivalence ([3] Theorem 8.3), which is Theorem 1 of [7].

For Tannaka theory we follow Joyal-Street [5] (for the original sources see the references therein). Their construction of the Hopf algebra $End^{\vee}(T)$ of endomorphisms of a finite dimensional K -vector space-valued functor T , can be developed for a \mathcal{V}_0 -valued functor, $\mathcal{X} \xrightarrow{T} \mathcal{V}_0 \subset \mathcal{V}$, where \mathcal{V} is any cocomplete monoidal closed category, and \mathcal{V}_0 a (small) full subcategory of objects with duals ([9], see appendix A). There is a lifting $\mathcal{X} \xrightarrow{\tilde{T}} Comod_0(End^{\vee}(T))$ into the category of $End^{\vee}(T)$ -comodules with underlying object in \mathcal{V}_0 . In

¹meaning, without recourse to change of base and other sophisticated tools of topos theory over an arbitrary base topos.

[5] they prove that in the case of vector spaces, under suitable conditions on \mathcal{X} and F , this lifting is an equivalence.

Recall that given a regular category \mathcal{C} we can consider the category $\mathcal{R}el(\mathcal{C})$ of relations in \mathcal{C} . There is a faithful functor (the identity on objects) $\mathcal{C} \rightarrow \mathcal{R}el(\mathcal{C})$, and any regular functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ has an extension $\mathcal{R}el(\mathcal{C}) \xrightarrow{\mathcal{R}el(F)} \mathcal{R}el(\mathcal{D})$.

The category $\mathcal{R}el = \mathcal{R}el(\mathcal{E}ns)$ is a full subcategory of the category $\mathcal{S}upp$ of sup-lattices, set $\mathcal{R}el = \mathcal{S}upp_0$. This determines the base $\mathcal{V}, \mathcal{V}_0$ of a Tannaka context. Furthermore, a localic group is the same thing as an idempotent Hopf algebra in the category $\mathcal{S}upp$ (see section 1).

Given the set-valued functor $\mathcal{E} \xrightarrow{F} \mathcal{E}ns$ of a Galois context, we associate a Tannakian context as follows:

$$\begin{array}{ccccc} \beta^G & \xleftarrow{\tilde{F}} & \mathcal{E} & \xrightarrow{\quad} & \mathcal{R}el(\mathcal{E}) & \xrightarrow{\tilde{T}} & \mathcal{C}omod_0(H) \\ & \searrow & \downarrow F & & \downarrow T & \swarrow & \\ & & \mathcal{E}ns & \longrightarrow & \mathcal{R}el = \mathcal{S}upp_0, & & \end{array}$$

where $G = \mathcal{A}ut(F)$, $H = \mathcal{E}nd^\vee(T)$, and $T = \mathcal{R}el(F)$.

In the case where F is the inverse image of a point of an atomic topos, we prove that \tilde{F} is an equivalence if and only if \tilde{T} is so (Theorem 5.1). The result is based in two theorems. First, we prove that for any localic group G , there is an isomorphism of categories $\mathcal{R}el(\beta^G) \cong \mathcal{C}omod_0(G)$ (Theorem 3.6). Second, we prove that the Hopf algebra $\mathcal{E}nd^\vee(T)$ is localic, and that there is an isomorphism $\mathcal{A}ut(F) \cong \mathcal{E}nd^\vee(T)$ (Theorem 4.13).

In particular, it follows that the fundamental (recognition) theorem of Tannaka theory holds in a concrete example completely different than the known cases (Theorem 5.2).

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1. BACKGROUND, TERMINOLOGY AND NOTATION

In this section we recall some facts on sup-lattices, locales and monoidal categories, and in this way we fix notation and terminology.

We will consider the monoidal category $Supp$ of *sup-lattices*, whose objects are posets S with arbitrary suprema \bigvee (hence infima \bigwedge , 0 and 1) and whose arrows are the suprema-preserving-maps. We call these arrows *linear maps*. We will write S also for the underlying set of the lattice. We have a free sup-lattice-functor $\ell : \mathcal{E}ns \rightarrow Supp$ mapping $X \mapsto \ell X = \mathcal{P}(X)$ (power set) and $X \xrightarrow{f} Y \mapsto \ell X \xrightarrow{\ell f} \ell Y$ (direct image). A linear map $\ell X \rightarrow \ell Y$ is the same thing as a relation $R \subset X \times Y$, and this identification is compatible with composition. In this way the category $\mathcal{R}el$ of sets with relations as morphisms is a full subcategory $\mathcal{R}el \hookrightarrow Supp$. We define $Supp_0$ as the full subcategory of $Supp$ of objects of the form ℓX . Thus, $\mathcal{R}el = Supp_0 \subset Supp$.

The *tensor product* of two sup-lattices S and T is the codomain of the universal bilinear map $S \times T \rightarrow S \otimes T$. Given $(s, t) \in S \times T$, we denote the corresponding element in $S \otimes T$ by $s \otimes t$. The unit for \otimes is the sup-lattice $2 = \{0 \leq 1\}$. The linear map $S \otimes T \xrightarrow{\psi} T \otimes S$ sending $s \otimes t \mapsto t \otimes s$ is a symmetry. It is immediate to check that $\ell X \otimes \ell Y = \ell(X \times Y)$.

Recall that, as in any monoidal category, a *duality* between two sup-lattices T and S is a pair of arrows $2 \xrightarrow{\eta} T \otimes S$, $S \otimes T \xrightarrow{\varepsilon} 2$ satisfying two triangular equations. We say in this case that T is *right dual* to S and denote $S^\wedge = T$. The arrows $2 \xrightarrow{\eta} \ell X \otimes \ell X$, $\ell X \otimes \ell X \xrightarrow{\varepsilon} 2$, defined on the generators as $\eta(1) = \bigvee_x x \otimes x$ and $\varepsilon(x, y) = \delta_{x=y}$ determine a duality, and in this way the objects of the form ℓX have duals and furthermore they are self-dual, $(\ell X)^\wedge = \ell X$. If $R \subset X \times Y$ is the relation corresponding to a linear map $\ell X \rightarrow \ell Y$, then the opposite relation $R^{op} \subset Y \times X$ corresponds to the dual map $(\ell Y)^\wedge \rightarrow (\ell X)^\wedge$.

As in any monoidal category, an *algebra* (or *monoid*) in $Supp$ is an object S with an associative multiplication $S \otimes S \xrightarrow{w} S$ which has a unit $2 \xrightarrow{u} S$. If w preserves the symmetry ψ , the algebra is commutative. An algebra morphism is a linear map which preserves w and u .

A locale is a sup-lattice S where finite infima \bigwedge distributes over arbitrary suprema \bigvee , that is, it is bilinear, and so induces a multiplication $\bigwedge : S \otimes S \rightarrow S$. A locale morphism is a linear map which preserves \bigwedge and 1. In this way locales are commutative algebras, and there is a full inclusion of categories $Loc \subset Alg_{Supp}$ into the category of commutative algebras in $Supp$.

1.1. In [7] locales are characterized as those commutative algebras such that $w(x \otimes x) = x$ and $u(1) = 1$.

A (commutative) Hopf algebra in $Supp$ is a group object in $(Alg_{Supp})^{op}$. A localic group (resp. monoid) G is a group (resp. monoid) object in the category Sp of localic spaces, which is defined to be the formal dual of the category of locales, $Sp = Loc^{op}$. Therefore G can be also considered as a Hopf algebra in $Supp$. The unit and the multiplication of G in Sp are the counit $G \xrightarrow{c} 2$ and comultiplication $G \xrightarrow{w} G \otimes G$ of a coalgebra structure for G in Alg_{Supp} . The inverse is an antipode $G \xrightarrow{\iota} G$. Morphisms correspond, and we actually have an equality of categories $Id-Hopf = Loc-Group$, between the category of idempotent Hopf algebras in $Supp$ and the category of localic groups.

2. PRELIMINARIES ON BIJECTIONS ON A LOCALE

As usual we view a relation λ between two sets X and Y as a map (i.e. function of sets) $X \times Y \xrightarrow{\lambda} 2$. We consider relations $X \times Y \xrightarrow{\lambda} G$ with values in an arbitrary locale G . Since $\ell(X \times Y) = \ell X \otimes \ell Y$, it follows that relations are the same thing that linear maps $\ell X \otimes \ell Y \xrightarrow{\lambda} G$. The results of this section will be used in this paper only in the case $X = Y$.

Consider the following formulae:

2.1. Axioms on a relation

$$\begin{array}{lll}
 ed) \bigvee_{y \in Y} \lambda\langle a, y \rangle = 1 & \text{for each } a & (\text{everywhere defined}) \\
 uv) \lambda\langle x, b_1 \rangle \wedge \lambda\langle x, b_2 \rangle = 0 & \text{for each } x, b_1 \neq b_2 & (\text{univalued}) \\
 su) \bigvee_{x \in X} \lambda\langle x, b \rangle = 1 & \text{for each } b & (\text{surjective}) \\
 in) \lambda\langle a_1, y \rangle \wedge \lambda\langle a_2, y \rangle = 0 & \text{for each } y, a_1 \neq a_2 & (\text{injective})
 \end{array}$$

Clearly any morphism of locales $G \rightarrow H$ preserves these four axioms.

A relation λ is a *function* if and only if satisfies axioms *ed)* and *uv)*. We say that a relation is an *op-function* when it satisfies axioms *su)* and *in)*. Then a relation is a *bijection* if and only if it is a function and an op-function.

2.2. Given two relations $X \times Y \xrightarrow{\lambda} G$, $X' \times Y' \xrightarrow{\lambda'} G$, the product relation $\lambda \boxtimes \lambda'$ is defined by the composition

$$\begin{aligned}
 X \times X' \times Y \times Y' &\xrightarrow{X \times \psi \times Y'} X \times Y \times X' \times Y' \xrightarrow{\lambda \times \lambda'} G \times G \xrightarrow{\wedge} G \\
 (\lambda \boxtimes \lambda')\langle (a, a'), (b, b') \rangle &= \lambda\langle a, b \rangle \wedge \lambda'\langle a', b' \rangle.
 \end{aligned}$$

The following is immediate and straightforward:

2.3. Proposition. *Each axiom in 2.1 for λ and λ' imply the respective axiom for the product $\lambda \boxtimes \lambda'$.* \square

2.4. Consider two maps $X \xrightarrow{f} X'$, $Y \xrightarrow{g} Y'$ and the following diagrams (we abuse notation by identifying X , ℓX and $(\ell X)^\wedge$. In this way, the inverse image of a map is the opposite relation, $f^{-1} = f^{op}$):

$$\begin{array}{ccc}
 \triangleright = \triangleright(f, g) & \diamond_1 = \diamond_1(f, g) & \diamond_2 = \diamond_2(f, g) \\
 \begin{array}{c} X \times Y \\ \downarrow f \times g \\ X' \times Y' \end{array} \xrightarrow{\lambda} G, & \begin{array}{ccc} X \times Y' & \xrightarrow{X \times g^{op}} & X \times Y \\ & \searrow f \times Y' & \downarrow \lambda \\ & X' \times Y' & \end{array} \xrightarrow{\lambda} G, & \begin{array}{ccc} X' \times Y & \xrightarrow{X' \times g} & X' \times Y' \\ & \searrow f^{op} \times Y & \downarrow \lambda' \\ & X \times Y & \end{array} \xrightarrow{\lambda} G
 \end{array}$$

expressing the equations $\triangleright : \lambda\langle a, b \rangle \leq \lambda'\langle f(a), g(b) \rangle$,

$$\diamond_1 : \lambda'\langle f(a), b' \rangle = \bigvee_{g(y)=b'} \lambda\langle a, y \rangle \quad \text{and} \quad \diamond_2 : \lambda'\langle a', g(b) \rangle = \bigvee_{f(x)=a'} \lambda\langle x, b \rangle.$$

2.5. Proposition. *If either \diamond_1 or \diamond_2 holds, then so does \triangleright .*

Proof. $\lambda\langle a, b \rangle \leq \bigvee_{g(y)=g(b)} \lambda\langle a, y \rangle = \lambda'\langle f(a), g(b) \rangle$ using \Diamond_1 . Clearly a symmetric arguing holds using \Diamond_2 . \square

We establish now under which hypothesis the reverse implication holds.

2.6. Proposition.

- 1) If λ and λ' are functions, then \triangleright implies \Diamond_1 . Thus, for functions \triangleright and \Diamond_1 are equivalent.
- 2) If λ and λ' are op-functions, then \triangleright implies \Diamond_2 . Thus, for op-functions \triangleright and \Diamond_2 are equivalent.
- 3) If λ and λ' are bijections, then \triangleright implies \Diamond_1, \Diamond_2 . Thus for bijections $\triangleright, \Diamond_1$ and \Diamond_2 are all three equivalent.

Proof. We prove 1), a symmetric proof yields 2), and 3) = 1) + 2).

$$\begin{aligned} \lambda'\langle f(a), b' \rangle &\stackrel{\lambda_{ed}}{=} \lambda'\langle f(a), b' \rangle \wedge \bigvee_y \lambda\langle a, y \rangle = \bigvee_y \lambda'\langle f(a), b' \rangle \wedge \lambda\langle a, y \rangle \stackrel{(*)}{=} \\ &\quad \bigvee_{g(y)=b'} \lambda'\langle f(a), b' \rangle \wedge \lambda\langle a, y \rangle \stackrel{\triangleright}{=} \bigvee_{g(y)=b'} \lambda\langle a, y \rangle, \end{aligned}$$

where for the equality marked with $(*)$ we used that if $g(y) \neq b'$ then $\lambda'\langle f(a), b' \rangle \wedge \lambda\langle a, y \rangle \stackrel{\triangleright}{\leq} \lambda'\langle f(a), b' \rangle \wedge \lambda'\langle f(a), g(y) \rangle \stackrel{\lambda'_{uv}}{=} 0$. \square

2.7. We generalize proposition 2.6 from maps $X \xrightarrow{f} X', Y \xrightarrow{g} Y'$ to ordinary relations of sets $R \subset X \times X', S \subset Y \times Y'$.

We consider the diagram $\Diamond = \Diamond(R, S)$:

$$\begin{array}{ccccc} & & X \times Y & & \\ & \nearrow^{X \times S^{op}} & & \searrow^{\lambda} & \\ X \times Y' & & \equiv & & G \\ & \searrow_{R \times Y'} & & \nearrow_{\lambda'} & \\ & & X' \times Y' & & \end{array}$$

expressing the equation $\bigvee_{(y, b') \in S} \lambda\langle a, y \rangle = \bigvee_{(a, x') \in R} \lambda'\langle x', b' \rangle$.

It is clear that diagrams \Diamond_1 and \Diamond_2 are particular cases of diagrams \Diamond (take $R = f, S = g$ and $R = f^{op}, S = g^{op}$).

Generalizing even more, we consider two spans, which induce relations that we also denote with the same letters,

$$\begin{array}{ccc} & R & \\ f \swarrow & & \searrow f' \\ X & & X' \end{array}, \quad \begin{array}{ccc} & S & \\ g \swarrow & & \searrow g' \\ Y & & Y' \end{array} \quad R = f' \circ f^{op}, \quad S = g' \circ g^{op},$$

provided with a relation $R \times S \xrightarrow{\theta} G$. Then:

2.8. Proposition. *Let R, S be any two spans as above such that $\Diamond_1(f', g')$ and $\Diamond_2(f, g)$ hold. Then $\Diamond(R, S)$ holds.*

Proof. We use the elevators calculus, see appendix B.

$$\begin{array}{c}
 \begin{array}{c} X & Y' \\ \parallel & \backslash \\ X & Y \\ \lambda & \searrow \\ & G \end{array} = \begin{array}{c} \begin{array}{c} X & Y' \\ \parallel & \backslash \\ X & S \\ \parallel & \backslash \\ X & Y \\ \lambda & \searrow \\ & G \end{array} \\ \begin{array}{c} \parallel & \backslash \\ X & Y \\ \lambda & \searrow \\ & G \end{array} \end{array} \stackrel{\Diamond_2}{=} \begin{array}{c} \begin{array}{c} X & Y' \\ \parallel & \backslash \\ X & S \\ \parallel & \backslash \\ R & S \\ \theta & \searrow \\ & G \end{array} \\ \begin{array}{c} \parallel & \backslash \\ X & Y \\ \lambda & \searrow \\ & G \end{array} \end{array} = \begin{array}{c} \begin{array}{c} X & Y' \\ \parallel & \backslash \\ R & Y' \\ \parallel & \backslash \\ R & S \\ \theta & \searrow \\ & G \end{array} \\ \begin{array}{c} \parallel & \backslash \\ X & Y \\ \lambda & \searrow \\ & G \end{array} \end{array} \stackrel{\Diamond_1}{=} \begin{array}{c} \begin{array}{c} X & Y' \\ \parallel & \backslash \\ R & Y' \\ \parallel & \backslash \\ R & Y' \\ \lambda' & \searrow \\ & G \end{array} \\ \begin{array}{c} \parallel & \backslash \\ X & Y \\ \lambda & \searrow \\ & G \end{array} \end{array} = \begin{array}{c} \begin{array}{c} X & Y' \\ \parallel & \backslash \\ X' & Y' \\ \parallel & \backslash \\ X' & Y' \\ \lambda' & \searrow \\ & G \end{array} \\ \begin{array}{c} \parallel & \backslash \\ X & Y \\ \lambda & \searrow \\ & G \end{array} \end{array}
 \end{array}$$

□

The role of diagram \triangleright generalizes to this situation as

$$(2.9) \quad \begin{array}{ccc} R \times S & \xrightarrow{\theta} & G \\ f \times g \downarrow & \geq & \uparrow \lambda \\ X \times Y & & \end{array}, \quad \begin{array}{ccc} R \times S & \xrightarrow{\theta} & G \\ f' \times g' \downarrow & \geq & \uparrow \lambda' \\ X' \times Y' & & \end{array}$$

For the remainder of this section, we will assume we have $R \times S \xrightarrow{\theta} G$ such that (2.9) holds. From propositions 2.6 and 2.8 we have:

2.10. Proposition. *Let R, S, θ as above, and λ, λ' bijections. Then, if θ is a bijection, \Diamond holds (it is enough that λ is an op-function and λ' is a function).* □

A converse of this proposition holds when the spans are relations, but it is convenient to prove the following lemma first:

2.11. Lemma. *If λ and λ' are functions, and θ satisfies uv), then equation 1) holds. Symmetrically, if λ and λ' are op-functions, and θ satisfies in), then equation 2) holds.*

$$\begin{aligned}
 1) \quad & \lambda\langle f(r), b \rangle \wedge \lambda'\langle f'(r), b' \rangle = \bigvee_{\substack{g(v)=b \\ g'(v)=b'}} \theta\langle r, v \rangle. \\
 2) \quad & \lambda\langle a, g(s) \rangle \wedge \lambda'\langle a', g'(s) \rangle = \bigvee_{\substack{f(u)=a \\ f'(u)=a'}} \theta\langle u, s \rangle.
 \end{aligned}$$

Proof. We only prove the first statement, since the second one clearly has a symmetric proof. Note that by proposition 2.6 equations $\Diamond_1(f, g)$ and $\Diamond_1(f', g')$ hold.

$$\begin{aligned}
 \lambda\langle f(r), b \rangle \wedge \lambda'\langle f'(r), b' \rangle & \stackrel{\Diamond_1}{=} \bigvee_{g(v)=b} \theta\langle r, v \rangle \wedge \bigvee_{g'(w)=b'} \theta\langle r, w \rangle = \\
 & = \bigvee_{\substack{g(v)=b \\ g'(w)=b'}} \theta\langle r, v \rangle \wedge \theta\langle r, w \rangle \stackrel{\theta_{uw}}{=} \bigvee_{\substack{g(v)=b \\ g'(v)=b'}} \theta\langle r, v \rangle.
 \end{aligned}$$

□

2.12. Proposition. *Let R, S be any two relations, with λ, λ' bijections. Then, if diagram \diamond holds, θ is a bijection.*

Proof. It is easy to check that the axioms uv) and in) for θ follow from the corresponding axioms for λ and λ' using (2.9) and the fact that the maps (f, f') and (g, g') are injective. We prove now the axiom ed) and axiom su) follows in a symmetrical way:

$$\begin{aligned} \bigvee_{v \in S} \theta \langle r, v \rangle &= \bigvee_{\substack{b \in Y \\ b' \in Y'}} \bigvee_{\substack{g(v)=b \\ g'(v)=b'}} \theta \langle r, v \rangle \stackrel{2.111)}{=} \bigvee_{\substack{b \in Y \\ b' \in Y'}} \lambda \langle fr, b \rangle \wedge \lambda' \langle f'r, b' \rangle = \\ &= \bigvee_{b \in Y} \lambda \langle fr, b \rangle \wedge \bigvee_{b' \in Y'} \lambda' \langle f'r, b' \rangle = 1 \wedge 1 = 1. \end{aligned}$$

□

2.13. Remark. Note that the diagrams \triangleright in 2.9 mean that $\theta \leq \lambda \boxtimes \lambda' \circ (f, f') \times (g, g')$ (see 2.2). In particular θ may be equal to this composition, $R \times S \xrightarrow{(f, f') \times (g, g')} X \times X' \times Y \times Y' \xrightarrow{\lambda \boxtimes \lambda'} G$.

We found it convenient to combine 2.10, 2.12 and 2.13 into:

2.14. Proposition. *Let R, S be any two relations, with λ, λ' bijections, and $\theta = \lambda \boxtimes \lambda' \circ (f, f') \times (g, g')$. Then, diagram \diamond holds if and only if θ is a bijection.* □

3. THE ISOMORPHISM $Comod_0(G) = Rel(\beta^G)$

The purpose of this section is to establish an isomorphism of categories between $Comod_0(G)$ and $Rel(\beta^G)$, where G is a fixed localic group, or, what amounts to the same thing, an idempotent Hopf algebra in the monoidal category $Supp$ of sup-lattices, as we explained in section 1.

3.1. Recall that given a regular category \mathcal{C} we can consider the category $Rel(\mathcal{C})$ of relations in \mathcal{C} . There is a faithful functor (the identity on objects) $\mathcal{C} \rightarrow Rel(\mathcal{C})$, and any regular functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ has an extension $Rel(\mathcal{C}) \xrightarrow{Rel(F)} Rel(\mathcal{D})$. It can be seen that F is an equivalence if and only if $Rel(F)$ is so.

3.2. The category $Comod_0(G)$.

As for any coalgebra, a comodule structure over G in $Supp$ is an object $S \in Supp$ together with a map $S \xrightarrow{\rho} G \otimes S$ preserving the counit and the comultiplication. A comodule morphism between two comodules is a map which makes the usual diagrams commute (see [5]). We define the category $Comod_0(G)$ as the full subcategory with objects the comodules of the form $S = \ell X$, for any set X . If we forget the comodule structure we have a faithful functor

$$Comod_0(G) \xrightarrow{T} Supp_0 = Rel.$$

3.3. The category β^G .

The construction of the category β^G of sets furnished with an action of G (namely, the *classifying topos* of G) requires some considerations (for

details see [3]). Given a set X , we can construct two locales $Rel(X)$ and $Aut(X)$ with a universal relation $X \times X \xrightarrow{j} Rel(X)$ and a universal bijection $X \times X \xrightarrow{\lambda} Aut(X)$, both in the sense of section 2, and universal in the category of locales (see [10], [3]). Clearly $Rel(X)$ is the free locale on $X \times X$. Then $Aut(X)$ is determined by the topology generated by the covers that force the four axioms in 2.1. Notice that it follows by definition that the points of the locales $Rel(X)$ and $Aut(X)$ are the relations and the bijections of the set X .

3.4. Remark. *There is a 1 to 1 correspondence between maps $X \times X \xrightarrow{\lambda} G$, linear maps $\ell X \otimes \ell X \xrightarrow{\lambda} G$ and locale morphisms $Rel(X) \xrightarrow{\lambda} G$. λ factors through an arrow $Aut(X) \xrightarrow{\lambda} G$ if and only if it is a bijection.* \square

Given $(x, y) \in X \times X$, we will denote $\langle x | y \rangle = j\langle x, y \rangle = \lambda\langle x, y \rangle$ indistinctly in both cases. We abuse notation and omit to indicate the associated sheaf morphism $Rel(X) \rightarrow Aut(X)$. The elements of the form $\langle x | y \rangle$ generate both locales by taking arbitrary suprema of finite infima.

By the universal property, the following arrows defined on the generators are well defined on $Rel(X)$ and $Aut(X)$:

$$\begin{aligned} m : X \times X &\longrightarrow Rel(X) \otimes Rel(X) \longrightarrow Aut(X) \otimes Aut(X) \\ m\langle x | y \rangle &= \bigvee_z \langle x | z \rangle \otimes \langle z | y \rangle, \\ e : X \times X &\longrightarrow 2, & e\langle x | y \rangle &= \delta_{x=y}. \\ \iota : X \times X &\longrightarrow Aut(X), & \iota\langle x | y \rangle &= \langle y | x \rangle, \end{aligned}$$

Clearly m and e determine a coalgebra structure on $Rel(X)$. We let the reader check that m, e and ι are bijections (in the sense of section 2) with values in $Aut(X) \otimes Aut(X)$, 2 and $Aut(X)$ respectively, making $Aut(X)$ a Hopf algebra.

An action of a localic group G in a set X is defined as a localic group morphism $G \xrightarrow{\hat{\mu}} Aut(X)$. This corresponds to a Hopf algebra morphism $Aut(X) \xrightarrow{\mu} G$, which is completely determined by its value on the generators, that is, a bijection $X \times X \xrightarrow{\mu} G$, that in addition satisfies

$$(3.5) \quad m\mu = (\mu \otimes \mu)m, \quad \mu\iota = \mu, \quad e\mu = e.$$

(the structures in both Hopf algebras are indicated with the same letters).

As we shall see in Proposition 3.9, the equation $\mu\iota = \mu$ follows from the other two. That is, any action of G viewed as a monoid is automatically a group action.

Given two objects $X, X' \in \beta^G$, a morphism between them is a function between the sets $X \xrightarrow{f} X'$ satisfying $\mu\langle a | b \rangle \leq \mu'\langle f(a) | f(b) \rangle$. Notice that this is a \triangleright diagram as in section 2.

If we forget the action we have a faithful functor $\beta^G \xrightarrow{F} \mathcal{E}ns$ (which is the inverse image of a point of the topos, see [3] Proposition 8.2). Thus, we

have a commutative square (see 3.1):

$$\begin{array}{ccc} \beta^G & \longrightarrow & \mathcal{R}el(\beta^G) \\ F \downarrow & & \downarrow \mathcal{R}el(F) \\ \mathcal{E}ns & \longrightarrow & \mathcal{R}el. \end{array}$$

We have the following theorem, that we will prove in the rest of this section.

3.6. Theorem. *There is an isomorphism of categories making the triangle commutative:*

$$\begin{array}{ccc} \text{Comod}_0(G) & \xrightarrow{=} & \mathcal{R}el(\beta^G) \\ & \searrow T & \swarrow \mathcal{R}el(F) \\ & \text{Supp}_0 = \mathcal{R}el. & \end{array}$$

The identification between relations $R \subset X \times X'$ and linear maps $\ell X \rightarrow \ell X'$ lifts to the upper part of the triangle. \square

Recall that since the functor F is the inverse image of a point, it follows that monomorphisms of G -sets are injective maps.

3.7. Proposition. *Let $f : X \rightarrow X'$ a morphism of G -sets. Then for each $a, b \in X$,*

$$\mu' \langle f(a) | f(b) \rangle = \bigvee_{f(x)=f(a)} \mu \langle a | x \rangle.$$

In particular, if f is a monomorphism, we have $\mu' \langle f(a) | f(b) \rangle = \mu \langle a | b \rangle$.

Proof. Since the actions are bijections, in particular functions, by proposition 2.6 the \triangleright diagram implies the \diamond_1 diagram. The statement follows by taking $(a, f(b)) \in X \times X'$. \square

Proposition 3.7 says that the subobjects $Z \hookrightarrow X$ of an object X in β^G are the subsets $Z \subset X$ such that the restriction of the action $Z \times Z \subset X \times X \xrightarrow{\mu} G$ is an action on Z . We have:

3.8. Proposition. *Let X be a G -set and $Z \subset X$ any subset. If the restriction of the action to Z is a bijection, then it is already an action.*

Proof. We have to check the equations in 3.5. The only one that requires some care is the first. By hypothesis (1) $m\mu \langle a | b \rangle = \bigvee_{x \in X} \mu \langle a | x \rangle \otimes \mu \langle x | b \rangle$.

We claim that when $a, b \in Z$, this equation still holds by restricting the supremum to the $x \in Z$. In fact, we have (2) $1 = \bigvee_{y, z \in Z} \mu \langle a | y \rangle \otimes \mu \langle z | b \rangle$.

Then, the claim follows by taking the infimum in both sides of equations (1) and (2). \square

3.9. Proposition. *Given a localic group G and a localic monoid morphism $G \xrightarrow{\hat{\mu}} \text{Rel}(X)$, there exists a unique action of G in X such that*

$$\begin{array}{ccc} \text{Rel}(X) & \xleftarrow{\hat{\mu}} & G, \\ & \nwarrow & \nearrow \hat{\mu} \\ & \text{Aut}(X) & \end{array} \quad \text{i.e.} \quad \begin{array}{ccc} \text{Rel}(X) & \xrightarrow{\mu} & G. \\ & \searrow & \nearrow \mu \\ & \text{Aut}(X) & \end{array}$$

Proof. μ is determined by a relation $X \times X \xrightarrow{\mu} G$ preserving m and e . It factorizes through $\text{Aut}(X)$ provided it is a bijection, and the factorization defines an action if it also preserves ι .

Consider the following commutative diagram

$$\begin{array}{ccccc} & & X \times X & \xrightarrow{m} & \text{Rel}(X) \otimes \text{Rel}(X) \\ & \swarrow e & \downarrow \mu & & \downarrow \mu \otimes \mu \\ 2 & \xleftarrow{e} & G & \xrightarrow{m} & G \otimes G \\ & \searrow u & & \downarrow \iota \otimes G & \downarrow G \otimes \iota \\ & & G & \xleftarrow{\wedge} & G \otimes G \end{array}$$

Chasing an element $(b, b) \in X \times X$ all the way down to G using the arrow $G \otimes \iota$ it follows $\bigvee_y \mu\langle b|y \rangle \wedge \iota\mu\langle y|b \rangle = 1$. Thus, in particular, we have

(1) $\bigvee_y \mu\langle b|y \rangle = 1$. Chasing in the same way an element (a, b) with $a \neq b$,

but this time using the arrow $\iota \otimes G$, it follows $\bigvee_x \iota\mu\langle a|x \rangle \wedge \mu\langle x|b \rangle = 0$. Thus

(2) $\iota\mu\langle a|x \rangle \wedge \mu\langle x|b \rangle = 0$ for all x .

We will see now that $\iota\mu \leq \mu\iota$ (since $\iota^2 = id$, it follows that also $\mu\iota \leq \iota\mu$).

$$\iota\mu\langle a|b \rangle \stackrel{(1)}{=} \iota\mu\langle a|b \rangle \wedge \bigvee_y \mu\langle b|y \rangle = \bigvee_y \iota\mu\langle a|b \rangle \wedge \mu\langle b|y \rangle \stackrel{(2)}{=} \iota\mu\langle a|b \rangle \wedge \mu\langle b|a \rangle, \text{ since}$$

all the other terms in the supremum are 0. Then $\iota\mu\langle a|b \rangle \leq \mu\langle b|a \rangle = \mu\iota\langle a|b \rangle$.

Thus we have $\iota\mu\langle a|b \rangle = \mu\iota\langle a|b \rangle (= \mu\langle b|a \rangle)$. With this, it is clear from the above that the rest of the statement follows. \square

3.10. Proposition. *There is a bijection between the objects of the categories $\text{Comod}_0(G)$ and $\text{Rel}(\beta^G)$.*

Proof. Since $(\ell X)^\wedge = \ell X$, we have a bijection of linear maps

$$\frac{\ell X \xrightarrow{\rho} G \otimes \ell X}{\ell X \otimes \ell X \xrightarrow{\mu} G}.$$

As with every duality (ε, η) , μ is defined as the composition

$$\mu : \ell X \otimes \ell X \xrightarrow{\rho \otimes \ell X} G \otimes \ell X \otimes \ell X \xrightarrow{G \otimes \varepsilon} G.$$

And conversely, we construct ρ as the composition

$$\rho : \ell X \xrightarrow{\ell X \otimes \eta} \ell X \otimes \ell X \otimes \ell X \xrightarrow{\mu \otimes \ell X} G \otimes \ell X.$$

It is easy to check that ρ preserves the counit and the comultiplication of G if and only if μ does. But by proposition 3.9, such a μ is an action $G \xrightarrow{\mu} \text{Aut}(X)$ (recall remark 3.4). \square

The product of two G -sets X and X' is equipped with the action given by the product relation $\mu \boxtimes \mu'$ (2.2), which is an action by proposition 2.3.

An arrow of the category $\text{Rel}(\beta^G)$ is a monomorphism $R \hookrightarrow X \times X'$, in particular, a relation of sets $R \subset X \times X'$. It follows from propositions 3.7 and 3.8, that a relation $R \hookrightarrow X \times X'$ in the category β^G is the same thing that a relation of sets $R \subset X \times X'$ such that the restriction of the product action to R is a bijection on R . The following proposition finishes the proof of theorem 3.6.

3.11. Proposition. *Let X, X' be any two G -sets, and $R \subset X \times X'$ a relation on the underlying sets. Then, R underlines a monomorphism of G -sets $R \hookrightarrow X \times X'$ if and only if the corresponding linear map $R : \ell X \rightarrow \ell X'$ is a comodule morphism.*

Proof. Let θ be the restriction of the product action $\mu \times \mu'$ to R . We claim that the diagram expressing that $R : \ell X \rightarrow \ell X'$ is a comodule morphism is equivalent to the diagram \diamond in 2.7. The proof follows then by proposition 2.14.

proof of the claim: It can be done by chasing elements in the diagrams, or more generally by using the elevators calculus explained in appendix B:

The comodule morphism diagram is the equality

$$(3.12) \quad \begin{array}{c} \ell X \\ \parallel \\ \ell X \end{array} \begin{array}{c} \nearrow \eta \\ \searrow \end{array} \begin{array}{c} \ell X \\ \searrow \mu_X \\ G \end{array} = \begin{array}{c} \ell X \\ \searrow R \\ \ell X' \end{array} \begin{array}{c} \nearrow \eta \\ \searrow \end{array} \begin{array}{c} \ell X' \\ \searrow \mu_{X'} \\ G \end{array},$$

while the diagram \diamond is

$$(3.13) \quad \begin{array}{c} \ell X \\ \parallel \\ \ell X \end{array} \begin{array}{c} \nearrow \eta \\ \searrow \end{array} \begin{array}{c} \ell X' \\ \parallel \\ \ell X' \end{array} = \begin{array}{c} \ell X \\ \searrow R \\ \ell X' \end{array} \begin{array}{c} \parallel \\ \ell X' \end{array}$$

$$\begin{array}{c} X \\ \diagup \eta \diagdown \\ X \quad Y \quad X \\ \parallel \quad \quad \quad \diagdown \varepsilon \diagup \\ X \end{array} = \begin{array}{c} X \\ \parallel \\ X \end{array} \quad \text{and} \quad \begin{array}{c} Y \\ \parallel \\ Y \quad X \quad Y \\ \quad \quad \quad \diagdown \varepsilon \diagup \\ \quad \quad \quad Y \end{array} = \begin{array}{c} Y \\ \parallel \\ Y. \end{array}$$
$$\begin{array}{c}
\begin{array}{ccccccc}
lX & & & lX' & & & \\
\parallel & & \nearrow \eta \searrow & \parallel & & & \\
lX & & lX & lX & lX' & & \\
\parallel & & \parallel & \searrow R/ & \parallel & & \\
lX & & lX & lX' & lX' & & \\
\searrow \mu_X/ & & \searrow \varepsilon/ & & & & \\
G & & & & & &
\end{array} \\
= \\
\begin{array}{ccccccc}
lX & & & lX' & & & \\
\parallel & & \nearrow \eta \searrow & \parallel & & & \\
lX & & lX & lX & lX' & & \\
\searrow \mu_X/ & & \searrow R/ & \parallel & & & \\
G & & lX' & lX' & & & \\
\searrow \varepsilon/ & & & & & &
\end{array}
\end{array}
\stackrel{(3.12)}{=}
\begin{array}{c}
\begin{array}{ccccccc}
lX & & & lX' & & & \\
\parallel & & \nearrow \eta \searrow & \parallel & & & \\
lX & & lX & lX & lX' & & \\
\searrow \mu_{X'}/ & & \searrow R/ & \parallel & & & \\
G & & lX' & lX' & & & \\
\searrow \varepsilon/ & & & & & &
\end{array} \\
=
\end{array}$$

[illegible]

$$\begin{aligned}
& \begin{array}{c} \ell X \\ \parallel \\ \ell X \end{array} \begin{array}{c} \diagup \eta \diagdown \\ \ell X \quad \ell X \end{array} \\
= & \begin{array}{c} \parallel \\ \ell X \end{array} \begin{array}{c} \parallel \\ \ell X \end{array} \begin{array}{c} \diagdown R \diagup \\ \ell X' \end{array} \\
& \begin{array}{c} \diagdown \mu_X \diagup \\ G \end{array} \begin{array}{c} \parallel \\ \ell X' \end{array} \begin{array}{c} \diagup \eta \diagdown \\ \ell X' \quad \ell X' \end{array} \\
& \begin{array}{c} \diagdown \varepsilon \diagup \\ \ell X' \end{array} \\
& \begin{array}{c} \ell X \\ \parallel \\ \ell X \end{array} \begin{array}{c} \diagup \eta \diagdown \\ \ell X \quad \ell X \end{array} \\
\stackrel{(\triangle)}{=} & \begin{array}{c} \parallel \\ \ell X \end{array} \begin{array}{c} \parallel \\ \ell X \end{array} \begin{array}{c} \diagdown R \diagup \\ \ell X' \end{array} \\
& \begin{array}{c} \diagdown \mu_X \diagup \\ G \end{array} \begin{array}{c} \parallel \\ \ell X' \end{array} \\
= & \begin{array}{c} \ell X \\ \parallel \\ \ell X \end{array} \begin{array}{c} \diagup \eta \diagdown \\ \ell X \quad \ell X \end{array} \\
& \begin{array}{c} \diagdown \mu_X \diagup \\ G \end{array} \begin{array}{c} \diagdown R \diagup \\ \ell X' \end{array}
\end{aligned}$$

□

4. THE GALOIS AND THE TANNAKIAN CONTEXTS

The Galois context.**4.1. The localic group of automorphisms of a functor.**

Given any small category \mathcal{C} and a functor $\mathcal{C} \xrightarrow{F} \mathcal{E}ns$, the localic group of automorphisms of F is defined as the locale which satisfies the following universal property:

For each $C \in \mathcal{C}$, there is a bijection $FC \times FC \xrightarrow{\lambda_C} Aut(F)$ such that for each $C \xrightarrow{f} C'$ in \mathcal{C} the diagram \triangleright holds, and for any other such data, there is a unique morphism of locales ϕ such that $\phi\lambda_C = \phi_C$, $\phi\lambda_{C'} = \phi_{C'}$, as indicated in the following diagram:

$$\begin{array}{ccccc}
 FC \times FC & & & & \\
 \downarrow F(f) \times F(f) & \searrow \lambda_C & & \searrow \phi_C & \\
 & & Aut(F) & \xrightarrow{\phi} & D. \\
 & \nearrow \lambda_{C'} & & \nearrow \phi_{C'} & \\
 FC' \times FC' & & & &
 \end{array}$$

It follows that a point $Aut(F) \rightarrow 2$ corresponds exactly to the data defining a natural isomorphism of F .

Given $(a, b) \in FC \times FC$, we will denote $\langle C, a|b \rangle = \lambda_C(a, b)$. The bijections λ_C determine a morphism of locales $Aut(FC) \xrightarrow{\mu_C} Aut(F)$, $\mu_C \langle a|b \rangle = \langle C, a|b \rangle$.

It is cumbersome but straightforward to check that for each $C \in \mathcal{C}$ the following three arrows are bijections:

$$FC \times FC \xrightarrow{m_C} Aut(F) \otimes Aut(F), \quad FC \times FC \xrightarrow{\iota_C} Aut(F), \quad FC \times FC \xrightarrow{e_C} 2.$$

$$m_C(a, b) = \bigvee_{x \in FC} \langle C, a|x \rangle \otimes \langle C, x|b \rangle, \quad \iota_C(a, b) = \langle C, b|a \rangle, \quad e_C(a, b) = \delta_{a=b}.$$

It follows that there is a localic group structure on $Aut(F)$ such that μ_C becomes an action of $Aut(F)$ on FC , and such that for any $C \xrightarrow{f} C' \in \mathcal{C}$, $F(f)$ is a morphism of actions. In this way there is a lifting \tilde{F} of the functor F into β^G , $\mathcal{C} \xrightarrow{\tilde{F}} \beta^G$, for $G = Aut(F)$.

We consider a connected atomic topos with a point $\mathcal{E}ns \xrightarrow{f} \mathcal{E}$, with inverse image $f^* = F$, $\mathcal{E} \xrightarrow{F} \mathcal{E}ns$. The full subcategory of connected objects $\mathcal{C} \subset \mathcal{E}$ furnished with the canonical topology is a small site for \mathcal{E} and satisfies the following axioms:

4.2. Assumption. Let \mathcal{C} be a small category and $\mathcal{C} \xrightarrow{F} \mathcal{E}$ a functor such that:

- i) Every arrow $C \rightarrow C'$ in \mathcal{C} is an strict epimorphism.
- ii) For every $C \in \mathcal{C}$, $FC \neq \emptyset$.
- iii) F preserves strict epimorphisms.

iv) The diagram of F, Γ_F^2 , is a cofiltered category.

From i) it easily follows that Γ_F is a poset ([3] proposition 6.2).

A small category \mathcal{C} and a functor $\mathcal{C} \xrightarrow{F} \mathcal{E}ns$ define a connected atomic site with a point precisely when this assumption is satisfied.

In [3] the following theorem is proved:

4.3. Theorem ([3] 8.3). *If $\mathcal{C} \xrightarrow{F} \mathcal{E}ns$ satisfy assumption 4.2, then \tilde{F} induces an equivalence of topoi, with inverse image denoted also $\tilde{F} : \mathcal{E} \xrightarrow{\cong} \beta^G$, where $\mathcal{E} = \tilde{\mathcal{C}}$ is the topos of sheaves on \mathcal{C} . \square*

By definition of $Aut(F)$, given $C \xrightarrow{f} C'$, if $a' = F(f)(a)$, $b' = F(f)(b)$, then $\langle C, a|b \rangle \leq \langle C', a'|b' \rangle$.

In [3] an explicit construction of $Aut(F)$ is given and the following key result of localic Galois Theory is proved (however, we do not need and will not use this deep result in this paper):

4.4. Theorem ([3]D1 6.9, 6.11).

- 1) For any $C \in \mathcal{C}$ and $(a, b) \in FC \times FC$, $\langle C, a|b \rangle \neq 0$.
- 2) Given any other $(a', b') \in FC' \times FC'$, if $\langle C, a|b \rangle \leq \langle C', a'|b' \rangle$, then there exists $C \xrightarrow{f} C'$ in \mathcal{C} such that $a' = F(f)(a)$, $b' = F(f)(b)$.

From this, Theorem 4.3 follows by a formal topos theoretic reasoning.

The Tannakian context.

For generalities concerning Tannaka theory see appendix A.

4.5. The Tannakian context associated to a locally connected topos.

We consider a connected locally connected topos with a point $\mathcal{E}ns \xrightarrow{f} \mathcal{E}$, with inverse image $f^* = F, \mathcal{E} \xrightarrow{F} \mathcal{E}ns$, and a small site of connected objects $\mathcal{C} \subset \mathcal{E}$. We have a diagram (see 3.1):

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathcal{R}el(\mathcal{E}) \\ F \downarrow & & \downarrow \mathcal{R}el(F) \\ \mathcal{E}ns & \longrightarrow & \mathcal{R}el. \end{array}$$

This determines a Tannakian context as in appendix A, with $\mathcal{X} = \mathcal{R}el(\mathcal{E})$, $\mathcal{V} = Supp$, $\mathcal{V}_0 = \mathcal{R}el = Supp_0$ and $T = \mathcal{R}el(F)$. Furthermore, in this case \mathcal{X} , \mathcal{V} and T are monoidal and symmetric, and every object of \mathcal{X} has a right dual. Thus, $End^\vee(T)$ (if it exists) is a (commutative) Hopf algebra (proposition A.5).

Note that on objects $TX = FX$, and on arrows in \mathcal{E} , $T(f) = F(f)$. $T(X \times Y) = TX \times TY$, if $X = \coprod_i C_i$, then $TX = \coprod_i TC_i$. If R is an arrow in $\mathcal{R}el(\mathcal{E})$, $T(R^{op}) = (TR)^{op}$. Given a connected component $C \hookrightarrow X$, to simplify language we will say that $TC \subset TX$ is a component of TX .

² Also called the *category of elements*.

The universal property which defines $End^\vee(T)$ is the following (note that the diagrams \triangleright and \diamond in section 2, and the corresponding equations, make sense for any sup-lattice):

$End^\vee(T)$ is a sup-lattice, for each $X \in \mathcal{X}$ there is a relation $TX \times TX \xrightarrow{\lambda_X} End^\vee(T)$ such that for each arrow $X \xrightarrow{R} X'$ in \mathcal{X} (i.e. relation in \mathcal{E}) the diagram $\diamond(TR, TR)$ holds, and for any other such data, there is a unique linear map ϕ such that $\phi\lambda_X = \phi_X$, $\phi\lambda_{X'} = \phi_{X'}$, as indicated in the following diagram:

$$\begin{array}{ccccc}
 & & TX \times TX & \xrightarrow{\phi_X} & \\
 & \nearrow^{TX \times TR^{op}} & & \searrow^{\lambda_X} & \\
 TX \times TX' & & \equiv & & End^\vee(T) \xrightarrow{\phi} H \\
 & \searrow_{TR \times TX'} & & \nearrow_{\lambda_{X'}} & \\
 & & TX' \times TX' & \xrightarrow{\phi_{X'}} &
 \end{array}$$

4.6. Notation. Given $(a, b) \in TX \times TX$, we will denote $[C, a, b] = \lambda_C \langle a, b \rangle$. We will also abbreviate $\diamond(R) = \diamond(TR, TR)$, and similarly for $\triangleright, \diamond_1$ and \diamond_2 .

We will see that the large coend which defines $End^\vee(T)$ exists by showing that the coend of T restricted to the (small) subcategory \mathcal{C} of connected objects can be (uniquely) furnished with relations λ_X for all objects $X \in \mathcal{X}$.

Consider more generally any sup-lattice H furnished with relations $TX \times TX \xrightarrow{\lambda_X} H$ such that all the $\diamond(R)$ diagrams commute. Then:

4.7. Proposition. Let $a, b \in TX$, if a and b are in different components of TX , then $\lambda_X \langle a, b \rangle = 0$. If they are in a same component TC , then $\lambda_X \langle a, b \rangle = \lambda_C \langle a, b \rangle$.

Proof. Assume $b \in TC$, and consider the inclusion $C \xhookrightarrow{i} X$. Then the $\diamond(i)$ square must commute:

$$\begin{array}{ccc}
 & TC \times TC & \\
 \nearrow^{TC \times T(i)^{op}} & & \searrow \\
 TX \times TC & & H \\
 \searrow_{TX \times T(i)} & & \nearrow \\
 & TX \times TX &
 \end{array}$$

If we chase an element $(a, b) \in TX \times TC$, we have $\lambda_X \langle a, b \rangle = 0$ if $a \notin TC$ and $\lambda_X \langle a, b \rangle = \lambda_C \langle a, b \rangle$ if $a \in TC$. \square

Assuming we only have λ_C for the connected C , it becomes evident that the only possible definition of λ_X for a general object X is, for $a, b \in TX$, if $a \in TC$ and $b \in TD$:

$$\lambda_X \langle a, b \rangle = \begin{cases} \lambda_C \langle a, b \rangle & \text{if } C = D, \\ 0 & \text{if } C \neq D. \end{cases}$$

It is immediate that with this definition the $\diamond(R)$ diagrams are still satisfied for every relation $R \hookrightarrow X \times X'$. It is also easy to prove that when H has a locale structure, if all the λ_C are bijections, so are all the λ_X . Thus:

4.8. Proposition. *Let H be any sup-lattice furnished with relations $TC \times TC \xrightarrow{\lambda_C} H$ for each $C \in \mathcal{C}$, and such that the $\diamond(R)$ diagrams commute for any relation $R \hookrightarrow C \times C'$ (resp. $\diamond_1(f)$ and $\diamond_2(f)$ diagrams commute for any arrow $C \xrightarrow{f} C'$). Then, H can be (uniquely) furnished with relations λ_X for all objects $X \in \mathcal{X}$ in such a way that the $\diamond(R)$ diagrams commute for any relation $R \hookrightarrow X \times X'$ (resp. $\diamond_1(f)$ and $\diamond_2(f)$ diagrams commute for any arrow $X \xrightarrow{f} X'$). Furthermore, when H has a locale structure, if all the λ_C are bijections, so are all the λ_X . \square*

In particular it follows that the suplattice $End^\vee(T)$ exists and that it is given by the coend of T restricted to the connected objects. In particular:

4.9. Remark. Every element in $End^\vee(T)$ is a supremum of elements of the form $[C, a, b]$ with C connected. \square

By the general Tannaka theory we know that $End^\vee(T)$ has a multiplication w and a unit u . The description of these arrows given below proposition A.4 yields in this case, for $X, Y \in \mathcal{X}$ (here, $F(1_C) = 1_{\mathcal{E}ns} = \{*\}$):

$$w([X, a, a'], [Y, b, b']) = [X \times Y, (a, b), (a', b')], \quad u(1) = [1_C, *, *].$$

This defines a locale structure where $U \wedge V = w(U, V)$ provided that for any element $U \in End^\vee(T)$, $w(U, U) = U$ and $U \leq u(1)$ (see 1.1). We claim this is the case. It is enough to check it for the generators:

Let $[X, a, b]$, consider the diagonal $X \xrightarrow{\Delta} X \times X$, the arrow $X \xrightarrow{\pi} 1$, and the following \diamond diagrams (where $H = End^\vee(T)$):

$$\begin{array}{ccc} & TX \times TX & \\ TX \times TX \times TX & \xrightarrow{TX \times \Delta^{op}} & TX \times TX \\ & \searrow \Delta \times TX \times TX & \swarrow \lambda_X \\ & TX \times TX \times TX \times TX & \\ & \xrightarrow{\lambda_{TX \times TX}} & H \end{array} \quad \equiv \quad \begin{array}{ccc} & TX \times TX & \\ TX \times 1 & \xrightarrow{TX \times \pi^{op}} & TX \times TX \\ & \searrow \pi \times 1 & \swarrow \lambda_X \\ & 1 \times 1 & \\ & \xrightarrow{\lambda_1} & H \end{array}$$

The claim follows by chasing (a, b, b) in the first diagram and $(a, *)$ in the second (recall that Δ is injective).

Thus, $End^\vee(T)$ is a locale with top element $[1_C, *, *]$ and infimum $[X, a, a'] \wedge [Y, b, b'] = [X \times Y, (a, b), (a', b')]$.

Warning. For a general \diamond -cone in $Supp$, $TX \times TX \xrightarrow{\lambda_X} H$, the definition $\lambda_X \langle a, a' \rangle \wedge \lambda_Y \langle b, b' \rangle = \lambda_{X \times Y} \langle (a, b), (a', b') \rangle$ is not necessarily well defined and does not determine any operation.

Case of an Atomic Topos.

In the rest of this section we assume Assumption 4.2.

4.10. Proposition. *The relations $TX \times TX \xrightarrow{\lambda_X} End^\vee(T)$ are bijections and for any arrow $X \xrightarrow{f} X'$ in \mathcal{E} the $\triangleright(f)$ diagram holds.*

Proof. Recall that both \diamond_1 and \diamond_2 are particular cases of \diamond , and that either one of these implies \triangleright (proposition 2.5). It remains to see that the λ_X are bijections. By the second statement in proposition 4.8 it suffices to consider $X = C$ connected.

Axioms uv) and in): Assume $[C, z, x] \wedge [C, z, y] = [C \times C, (z, z), (x, y)] \neq 0$. Then, $(z, z), (x, y)$ must be in a same component TD of $TC \times TC$, $D \xrightarrow{(f, g)} C \times C$. Let $a, b \in TD$, $T(f)(a) = z$, $T(g)(a) = z$, and $T(f)(b) = x$, $T(g)(b) = y$. Since the diagram of $T (= F \text{ on } \mathcal{C})$ is a poset (Assumption 4.2), it follows that $f = g$. Thus $x = y$. Axiom in) follows in the same way.

Axioms ed) and su): $\bigvee_{x \in TC} [C, c, x] = 1$. It suffices to see that for every connected D and $a, b \in TD$, $[D, a, b] \leq \bigvee_{x \in TC} [C, c, x]$. By Assumption 4.2

take $(E, w) \xrightarrow{f} (C, c)$ in the diagram of $T (= F \text{ on } \mathcal{C})$. Then, $[D, a, b] = [D, T(g)(w), b] \stackrel{\diamond_1(g)}{=} \bigvee_{T(g)(y)=b} [E, w, y] \stackrel{\triangleright(f)}{\leq} \bigvee_{T(g)(y)=b} [C, T(f)(w), T(f)(y)] \leq \bigvee_x [C, c, x]$. Clearly a symmetric proof using $\diamond_2(f)$ and $\triangleright(g)$ shows su). \square

We now turn to the locale $Aut(F)$ of 4.1.

4.11. Proposition. *The locale $Aut(F)$ can be (uniquely) furnished with bijections $TX \times TX \xrightarrow{\lambda_X} Aut(F)$ for each $X \in \mathcal{X}$, and such that the $\diamond(R)$ diagram commutes for any relation $R \hookrightarrow X \times X'$.*

Proof. First, by Proposition 2.6 the $\diamond_1(f)$ and $\diamond_2(f)$ diagrams commute for all arrows $C \xrightarrow{f} C'$. Then, proposition 4.8 yields bijections $TX \times TX \xrightarrow{\lambda_X} Aut(F)$ such that the diagrams $\diamond_1(f)$ and $\diamond_2(f)$ commute for all arrows $X \xrightarrow{f} X'$. Now, given any relation $R \hookrightarrow X \times X'$, use proposition 2.10 with $\lambda = \lambda_X$, $\lambda' = \lambda_{X'}$, and $\theta = \lambda_R$, to prove that the $\diamond(R)$ diagram commute. \square

From proposition 4.10 it follows there is a unique morphism of locales $Aut(F) \xrightarrow{\rho} End^\vee(T)$ such that $\lambda_X \rho = \lambda_X$ for all X , and from Proposition 4.11 that there is a unique linear map $End^\vee(T) \xrightarrow{\sigma} Aut(F)$ such that $\lambda_X \sigma = \lambda_X$. By the uniqueness of the universal property we have $\rho \sigma = id_{End^\vee(T)}$. The other isomorphism equation $\sigma \rho = id_{Aut(F)}$ will follow if σ is a morphism of locales. But this is the case:

4.12. Proposition. *The linear map $End^\vee(T) \xrightarrow{\sigma} Aut(F)$ preserves infimum and 1, thus it is a locale morphism.*

Proof. In both locales the λ_X are bijections, in particular λ_1 satisfies ed). It follows then that in both locales $\lambda_1(*, *) = 1$. Since $\sigma \lambda_1 = \lambda_1$, this shows that σ preserves 1. To show that infima are preserved, by the remark 4.9, it suffices to show that infima of the form $\lambda_C \langle c, c' \rangle \wedge \lambda_D \langle a, a' \rangle$ (with C, D connected) are preserved. As in Proposition 4.10 take $(E, w) \xrightarrow{f} (C, c)$ Then,

$(E, w) \xrightarrow{g} (D, a)$.

by lemma 2.11 with $\lambda = \lambda_C$, $\lambda' = \lambda_D$, and $\theta = \lambda_E$, it follows that the equation $\lambda_C\langle c, c' \rangle \wedge \lambda_D\langle a, a' \rangle = \bigvee_{\substack{T(f)(x)=c' \\ T(g)(x)=a'}} \lambda_E\langle w, x \rangle$ holds in both locales. The

proof finishes recalling that σ preserves suprema and all the λ_X . \square

We have finished the proof of the following theorem:

4.13. Theorem. *Given a pointed connected atomic topos satisfying 4.2, there is an isomorphism of locales $\text{End}^\vee(T) \cong \text{Aut}(F)$ commuting with the bijections λ_X , and the elements of the form $\lambda_C\langle a, b \rangle$ with C connected are sup-lattice generators.* \square

5. THE MAIN THEOREMS

A connected atomic topos with a point $\mathcal{E}ns \xrightarrow{f} \mathcal{E}$, with inverse image $f^* = F$, $\mathcal{E} \xrightarrow{F} \mathcal{E}ns$, determines a situation described in the following diagram:

$$\begin{array}{ccccccc}
 \beta^G & \longrightarrow & \mathcal{R}el(\beta^G) & \xrightarrow{=} & \text{Comod}_0(G) & \xrightarrow{=} & \text{Comod}_0(H) \\
 & \nwarrow \tilde{F} & & \nwarrow \mathcal{R}el(\tilde{F}) & & \nearrow \tilde{T} & \\
 & & \mathcal{E} & \longrightarrow & \mathcal{R}el(\mathcal{E}) & & \\
 & \searrow F & \downarrow F & & \downarrow T & \nearrow & \\
 & & \mathcal{E}ns & \longrightarrow & \mathcal{R}el \cong \text{Supp}_0 \subset \text{Supp} & &
 \end{array}$$

where $G = \text{Aut}(F)$, $T = \mathcal{R}el(F)$, $H = \text{End}^\vee(T)$ and the two isomorphisms in the first row of the diagram are given by Theorems 3.6 and 4.13.

It follows (see 3.1):

5.1. Theorem. *The (Galois) lifting functor \tilde{F} is an equivalence if and only if the (Tannaka) lifting functor \tilde{T} is such.* \square

In [3] it is proved that \tilde{F} is an equivalence (see 4.3), so we have:

5.2. Theorem. *The (Tannaka) lifting functor is an equivalence.* \square

APPENDIX A. TANNAKA THEORY

The Hopf algebra of automorphisms of a \mathcal{V} -functor.

Let \mathcal{V} be a cocomplete monoidal closed category with tensor product \otimes , unit object I and internal hom-functor hom . By definition for every object $V \in \mathcal{V}$, $\text{hom}(V, -)$ is right adjoint to $(-) \otimes V$. That is, for every X, Y , $\text{hom}(X \otimes V, Y) = \text{hom}(X, \text{hom}(V, Y))$. A pairing between two objects V, W is a pair of arrows $W \otimes V \xrightarrow{\varepsilon} I$ and $I \xrightarrow{\eta} V \otimes W$ satisfying the usual triangular equations. We say that W is the *left* dual of V , and denote $W = V^\vee$, and that V is *right* dual of W and denote $V = W^\wedge$. The following are basic equations:

$$V = V^{\vee\wedge}, \quad V = V^{\wedge\vee}, \quad \text{hom}(V, W) = W \otimes V^\vee, \quad \text{hom}(V^\wedge, W) = W \otimes V.$$

If V has a left dual, then $V^\vee = \text{hom}(V, I)$.

Recall that the object of natural transformations between \mathcal{V} -valued functors L, T , is given, if it exists, by the following end

$$(A.1) \quad Nat(L, T) = \int_{\mathcal{C}} hom(LC, TC).$$

We let $\mathcal{V}_0 \subset \mathcal{V}$ be a full subcategory such that all its objects have a right dual.

Let \mathcal{X} be a \mathcal{V} -category such that for any two functors $\mathcal{X} \xrightarrow{L} \mathcal{V}$ and $\mathcal{X} \xrightarrow{T} \mathcal{V}_0$ the coend in the following definition exists in \mathcal{V} (for example, if \mathcal{X} is small). Then, we define (in Joyal's terminology) the *Nat predual* as follows:

$$(A.2) \quad Nat^{\vee}(L, T) = \int^{\mathcal{C}} LC \otimes (TC)^{\wedge}.$$

Given $V \in \mathcal{V}$, recall that there is a functor $\mathcal{X} \xrightarrow{V \otimes T} \mathcal{V}$ defined by $(V \otimes T)(C) = V \otimes TC$. We have:

A.3. Proposition. *Given $T \in \mathcal{V}_0^{\mathcal{C}}$, we have a \mathcal{V} -adjunction*

$$\mathcal{V}^{\mathcal{C}} \begin{array}{c} \xrightarrow{Nat^{\vee}(-, T)} \\ \perp \\ \xleftarrow{(-) \otimes T} \end{array} \mathcal{V}.$$

Proof.

$$\begin{aligned} hom(Nat^{\vee}(L, T), V) &= hom\left(\int^{\mathcal{C}} LC \otimes TC^{\wedge}, V\right) = \int_{\mathcal{C}} hom(LC \otimes TC^{\wedge}, V) = \\ &= \int_{\mathcal{C}} hom(LC, hom(TC^{\wedge}, V)) = \int_{\mathcal{C}} hom(LC, V \otimes TC) = Nat(L, V \otimes T). \quad \square \end{aligned}$$

In particular we have that the end $Nat(L, T)$ exists and $Nat(L, T) = hom(Nat^{\vee}(L, T), I)$. It follows that $Nat^{\vee}(L, T)$ classifies natural transformations $L \Rightarrow T$ in the sense that they correspond to arrows $Nat^{\vee}(L, T) \rightarrow I$ in \mathcal{V} . This does not mean that $Nat(L, T)$ is the left dual of $Nat^{\vee}(L, T)$, which in general will not have a left dual. Passing from $Nat^{\vee}(L, T)$ to $Nat(L, T)$ loses information.

The unit of the adjunction $L \xRightarrow{\eta} Nat^{\vee}(L, T) \otimes T$ is a coevaluation, and if $\mathcal{X} \xrightarrow{H} \mathcal{V}_0$, it induces (in the usual manner) a cocomposition $Nat^{\vee}(L, H) \xrightarrow{\Delta} Nat^{\vee}(L, T) \otimes Nat^{\vee}(T, H)$. There is a counit $Nat^{\vee}(T, T) \xrightarrow{\varepsilon} I$ determined by the arrows $TC \otimes TC^{\vee} \xrightarrow{\varepsilon} I$ of the duality. All the preceding means exactly that the functors $\mathcal{X} \rightarrow \mathcal{V}_0$ are the objects of a \mathcal{V} -cocategory.

We define $End^{\vee}(T) = Nat^{\vee}(T, T)$, which is a coalgebra in \mathcal{V} . The coevaluation in this case becomes a $End^{\vee}(T)$ -comodule structure $TC \xrightarrow{\eta} End^{\vee}(T) \otimes TC$ on TC . In this way there is a lifting of the functor T into $Comod_0(H)$, $\mathcal{X} \xrightarrow{\tilde{T}} Comod_0(H)$, for $H = End^{\vee}(T)$, and $Comod_0(H)$ the full subcategory of comodules with underlying object in \mathcal{V}_0 .

A.4. Proposition. *If \mathcal{X} and T are monoidal, and \mathcal{V} has a symmetry, then $End^\vee(T)$ is a bialgebra. If in addition \mathcal{X} has a symmetry and T respects it, $End^\vee(T)$ is commutative (as an algebra). \square*

We will not prove this proposition here (see [5] for the case of vector spaces, and [9] for the general case), but show how the multiplication and the unit are constructed. The multiplication $End^\vee(T) \otimes End^\vee(T) \xrightarrow{w} End^\vee(T)$ is induced by the composites

$$w_{X,Y} : TX \otimes TX^\wedge \otimes TY \otimes TY^\wedge \xrightarrow{\cong} T(X \otimes Y) \otimes T(X \otimes Y)^\wedge \xrightarrow{\lambda_{X \otimes Y}} End^\vee(T).$$

The unit is given by the composition

$$u : I \rightarrow I \otimes I^\wedge \xrightarrow{\cong} T(I_{\mathcal{X}}) \otimes T(I_{\mathcal{X}})^\wedge \xrightarrow{\lambda_{I_{\mathcal{X}}}} End^\vee(T).$$

A.5. Proposition. *If in addition to the hypothesis of A.4 every object of \mathcal{X} has a right dual, then $End^\vee(T)$ is a Hopf algebra. \square*

The antipode $End^\vee(T) \xrightarrow{a} End^\vee(T)$ is induced by the composites

$$a_X : TX \otimes TX^\wedge \xrightarrow{\cong} T(X^\wedge) \otimes TX \xrightarrow{\lambda_{X^\wedge}} End^\vee(T).$$

APPENDIX B. ELEVATORS CALCULUS

This is a graphic notation invented by the first author in 1969 (which has remained for private draft use for understandable typographical reasons) to write equations in monoidal categories, ignoring associativity and suppressing the tensor symbol \otimes and the neutral object I . Arrows are written as cells, the identity arrow as a double line, and the symmetry as crossed double lines. This notation exhibits clearly the permutation associated to a composite of different symmetries, allowing to see if any two composites are the same simply by checking that they codify the same permutation³. Compositions are read from top to bottom.

Given arrows $f : C \rightarrow D$, $f' : C' \rightarrow D'$, the bifunctionality of the tensor product is the basic equality:

$$(B.1) \quad \begin{array}{c} C \quad C' \\ \diagdown \quad \parallel \\ f \quad \parallel \\ \diagup \quad \parallel \\ D \quad C' \\ \parallel \quad \diagdown \\ \parallel \quad f' \\ \diagup \quad \parallel \\ D \quad D' \end{array} = \begin{array}{c} C \quad C' \\ \parallel \quad \diagdown \\ \parallel \quad f' \\ \diagup \quad \parallel \\ C \quad D' \\ \diagdown \quad \parallel \\ f \quad \parallel \\ \diagup \quad \parallel \\ D \quad D' \end{array} = \begin{array}{c} C \quad C' \\ \diagdown \quad \diagdown \\ f \quad f' \\ \diagup \quad \diagup \\ D \quad D' \end{array}.$$

This allows to move cells up and down when there are no obstacles, as if they were elevators.

³ This is justified by a simple coherence theorem for symmetrical categories ([9] Proposition 2.3), particular case of [6] Corollary 2.2 for braided categories.

The naturality of the symmetry is the basic equality:

$$(B.2) \quad \begin{array}{c} C \quad C' \\ \diagdown \quad \diagup \\ f \quad \parallel \\ D \quad C' \\ \parallel \quad \diagdown \\ D \quad D' \\ \diagup \quad \diagdown \\ D' \quad D \end{array} = \begin{array}{c} C \quad C' \\ \diagdown \quad \diagup \\ f \quad \parallel \\ D \quad C' \\ \diagup \quad \diagdown \\ C' \quad D \\ \diagdown \quad \diagup \\ D' \quad D \end{array} = \begin{array}{c} C \quad C' \\ \diagup \quad \diagdown \\ \parallel \quad \diagup \\ C' \quad C \\ \parallel \quad \diagdown \\ C' \quad D \\ \diagdown \quad \diagup \\ D' \quad D \end{array}$$

Cells going up or down pass through symmetries by changing the column.

Combining the basic moves (B.1) and (B.2) we form configurations of cells that fit valid equations in order to prove new equations.

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